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An n-Dimensional Borg–Levinson Theorem

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Abstract. We show that the potential q is uniquely determined by the spectrum, and boundary values of the normal derivatives of the eigenfunctions of the Schrödinger operator $-\Delta+q$ with Dirichlet boundary conditions on a bounded domain Ω in $\mathbb{R}^n$. This and related results can be viewed as a direct generalization of the theorem in the title, which states that the spectrum and the norming constants determine the potential in the one dimensional case.

1. Introduction

Let $q(x)$ be a real-valued potential in $L^\infty[0,1]$ and let $y(x,\mu)$ solve the initial value problem

$$\begin{aligned} -y''+qy&=\mu y \quad \text{for} \quad x\in(0,1),\\ y(0,\mu)&=0,\\ y'(0,\mu)&=1. \end{aligned}$$

Define the sequence $\{\mu_i(q)\}_{i=1}^{\infty}$ of Dirichlet eigenvalues by the condition

$$y(1,\mu_i)=0$$

and define the norming constants c_i by

$$c_i(q)=\int_0^1 y^2(x,\mu_i)dx.$$

A well known result of Borg [B] and Levinson [L] is

Theorem 1.1. *Suppose that $q_1,q_2,\in L^\infty(0,1)$, are real-valued and that, for all i*

$$\mu_i(q_1)=\mu_i(q_2)$$

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and

$$c_i(q_1) = c_i(q_2);$$

then

$$q_1 = q_2.$$

It is possible to paraphrase Theorem 1.1 by

Corollary 1.2. *Suppose that $q_1, q_2 \in L^\infty(0,1)$, are real-valued and that, for all i*

$$\mu_i(q_1) = \mu_i(q_2),$$

and

$$y'(1, \mu_i(q_1); q_1) = y'(1, \mu_i(q_2); q_2),$$

then

$$q_1 = q_2.$$

Proof. Integrating the identity

$$\left(y \frac{\partial y'}{\partial \mu} - \frac{\partial y}{\partial \mu} y' \right)' = -y^2,$$

and setting $\mu = \mu_i$ yields the well known formula

$$c_i = \int_0^1 y^2(x, \mu_i)\, dx = \frac{\partial y}{\partial \mu}(1, \mu_i) y'(1, \mu_i). \tag{1.1}$$

As a function of μ, $y(1,\mu)$ is entire and of order 1/2 so that

$$y(1,\mu) = \prod_{i=1}^{\infty} \left(1 - \frac{\mu}{\mu_i} \right).$$

We may conclude from our hypothesis then that

$$y(1, \mu; q_1) = y(1, \mu; q_2),$$

and therefore that

$$\frac{\partial y}{\partial \mu}(1, \mu_i; q_1) = \frac{\partial y}{\partial \mu}(1, \mu_i; q_2),$$

and finally from (1.1), that

$$c_i(q_1) = c_i(q_2),$$

so that the corollary follows from Theorem 1.1 ■

Now, Corollary 1.2 has a direct generalization to higher dimensions; let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary and let $q(x) \in L^\infty(\Omega)$. Let $\{\mu_i(q)\}_{i=1}^{\infty}$ denote the eigenvalues of

$$-\Delta u + qu = \mu u \quad \text{in} \quad \Omega$$
$$u|_{\partial\Omega} = 0, \tag{1.2}$$

and let $\{\varphi_i(x)\}_{i=1}^{\infty}$ be a corresponding complete set of orthonormal eigenfunctions[1], then we have

Theorem 1.3. *Let $q_1, q_2 \in C^{\infty}(\bar{\Omega})$ be real-valued and suppose that, for each i*

$$\mu_i(q_1) = \mu_i(q_2)$$

and

$$\frac{\partial \varphi_i}{\partial \nu}(x; q_1) = \frac{\partial \varphi_i}{\partial \nu}(x; q_2) \quad \text{for all} \quad x \in \partial\Omega^1; \tag{1.3}$$

then

$$q_1(x) = q_2(x) \quad \text{for all} \quad x \in \Omega.$$

We may also consider different boundary conditions.

If $\{\lambda_i(q)\}_{i=1}^{\infty}$ denote the eigenvalues and $\{\psi_i(x; q)\}_{i=1}^{\infty}$ a complete set of orthonormal eigenfunctions of

$$-\Delta u + qu = \lambda u, \quad \frac{\partial u}{\partial \nu} + \alpha u|_{\partial\Omega} = 0, \tag{1.4}$$

where $\alpha(x)$ is a fixed smooth real-valued function on $\partial\Omega$, we have

Theorem 1.4. *Let $q_1, q_2 \in C^{\infty}(\bar{\Omega})$ be real-valued and suppose that, for each i,*

$$\begin{aligned} \lambda_i(q_1) &= \lambda_i(q_2), \\ \psi_i(x; q_1) &= \psi_i(x; q_2) \quad \text{for all} \quad x \in \partial\Omega^1; \end{aligned} \tag{1.5}$$

then

$$q_1(x) = q_2(x) \quad \text{for all} \quad x \in \Omega.$$

The bulk of the paper is devoted to the proof of Theorems 1.3 and 1.4; to this end we shall make use of the Dirichlet to Neumann map, which we define as follows: suppose that zero is not an eigenvalue of (1.2) and let u solve

$$-\Delta u + qu = 0 \quad \text{in} \quad \Omega, \tag{1.6}$$

$$u|_{\partial\Omega} = f, \tag{1.7}$$

we define

$$\Lambda_q f = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}. \tag{1.8}$$

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[1] To each eigenvalue μ_i we should properly associate not an eigenfunction but an eigenspace $V_i \subset L^2(\Omega)$; if $\varphi \in V_i$, then $\varphi \in C^1(\bar{\Omega})$, hence $W_i = \{f \mid f = (\partial\varphi/\partial\nu)|_{\partial\Omega} : \varphi \in V_i\}$ is a subspace of $L^2(\partial\Omega)$, equipped with the inner product $\langle \partial\varphi/\partial\nu|_{\partial\Omega}, \partial\psi/\partial\nu|_{\partial\Omega} \rangle = \langle \varphi, \psi \rangle_{L^2(\Omega)}$. Condition (1.3) should actually read

$$W_i(q_1) = W_i(q_2)$$

as inner product spaces, and (1.5) should read similarly.

If zero is not an eigenvalue of (1.4), let u solve (1.6) and replace (1.7) with

$$\frac{\partial u}{\partial \nu} + \alpha u|_{\partial\Omega} = g$$

to define

$$R_q g = u|_{\partial\Omega}. \tag{1.9}$$

Although R_q depends on $\alpha(x)$—which is known and fixed throughout—we do not indicate the dependence explicitly. If we let $\lambda \in \mathbb{C}$ we may replace q in (1.6) with $q-\lambda$ and consider $\Lambda_{q-\lambda}$ and $R_{q-\lambda}$ as functions of λ; we note that $\Lambda_{q-\lambda}$ and $R_{q-\lambda}$ are meromorphic operator-valued functions of λ (with poles exactly on the spectrum of the associated Schrödinger operators). We shall obtain both Theorem 1.3 and Theorem 1.4 as corollaries to

Theorem 1.5. *Let $q_1, q_2 \in L^\infty(\Omega)$ and suppose that, as meromorphic functions of $\lambda \in \mathbb{C}$, either*

$$R_{q_1-\lambda} = R_{q_2-\lambda}, \tag{1.10}$$

or

$$\Lambda_{q_1-\lambda} = \Lambda_{q_2-\lambda}, \tag{1.11}$$

then

$$q_1 = q_2.$$

To see, formally, the connection between Theorem 1.4 and Theorem 1.5, let $G(x,y,\lambda)$ be the Green's function for $-\Delta + q - \lambda$ with the boundary conditions (1.4); then the solution to

$$-\Delta u + (q-\lambda)u = 0, \quad \frac{\partial u}{\partial \nu} + \alpha u|_{\partial\Omega} = g \tag{1.12}$$

is given by

$$u(x) = \int_{\partial\Omega} G(x,y,\lambda) g(y) dS(y) \quad \text{for} \quad x \text{ in } \Omega,$$

while G is given by the eigenfunction expansion

$$G(x,y,\lambda) = \sum_{i=1}^{\infty} \frac{\psi_i(x)\psi_i(y)}{\lambda_i - \lambda}$$

so that, if we let x approach the $\partial\Omega$

$$R_{q-\lambda}(g) = \sum_{i=1}^{\infty} \psi_i(x)\Big|_{\partial\Omega} \frac{1}{\lambda_i - \lambda} \int_{\partial\Omega} \psi_i(y) g(y) dS(y),$$

which expresses $R_{q-\lambda}$ in terms of λ_i and $\psi_i|_{\partial\Omega}$ thus (formally) proving Theorem 1.4.

The last theorem we state is a sharpening of Theorem 1.5 in dimensions $n \geq 3$.

Theorem 1.6. *Let $q_1, q_2 \in L^\infty(\Omega)$, $n \geq 3$, and suppose that λ_0 is not a Dirichlet eigenvalue of q_1 or q_2. If*

$$\Lambda_{q_1-\lambda_0} = \Lambda_{q_2-\lambda_0},$$

then

$$q_1 = q_2.$$

For smooth potentials Theorem 1.6 is a direct consequence of a theorem in [S-U-II]; we include a somewhat simpler proof here, however. We also note that Theorem 1.6 is known to be true in dimension $n = 2$, provided that q_1 and q_2 are sufficiently close to constants (see [S-U, I]).

The paper is organized into three sections; in Sect. 2 we prove Theorems 1.5 and 1.6, and in Sect. 3 we use Theorem 1.5 to prove Theorems 1.3 and 1.4.

2. Proof of Theorems 1.5 and 1.6

We begin this section by constructing special solutions to (1.6) and (1.12), which shall be used to prove Theorems 1.5 and 1.6. We shall find these solutions by solving an equation in $\mathbb{R}^n$; in order to do this, we shall extend the potential $q(x)$ to be zero outside the domain Ω. We shall make use of the norm

$$\|\psi\|_{L^2_\delta} = \|(1+|x|)^\delta \psi\|_{L^2},$$

and the seminorm

$$|\psi|_s = \left(\lim_{R\to\infty} \frac{1}{R} \int_{|x|\leq R} |\psi|^2 \right)^{1/2}.$$

We shall need solutions to

$$-\Delta u + qu - \lambda u = 0 \tag{2.1}$$

of the form

$$u = e^{ik\cdot x} + \psi, \tag{2.2}$$

where

$$k\cdot k = \lambda; \quad k \in \mathbb{R}^n, \tag{2.3}$$

$$\psi, \nabla\psi \in L^2_\delta; \quad \delta < -\tfrac{1}{2}, \tag{2.4}$$

and satisfies

$$-\Delta\psi + q\psi - \lambda\psi = -qe^{ik\cdot x}; \tag{2.5}$$

in addition, ψ is a λ-outgoing solution to (2.5); that is

$$\left| \frac{\partial}{\partial r}\psi - i\sqrt{\lambda}\psi \right|_s = 0. \tag{2.6}$$

We summarize in a lemma.

Lemma 2.1. *For $\delta < -\frac{1}{2}$, there exists $\varepsilon(\delta) > 0$ such that if*

$$\|q(x)(1+|x|)^{-2\delta}\|_{L^\infty} < \varepsilon(\delta)\sqrt{\lambda}, \tag{2.7}$$

there exists a unique solution u to (2.1) of the form (2.2) such that ψ satisfies (2.4) and (2.6). In addition

$$\|\psi\|_{L^2_\delta} \leqq \frac{C(\varepsilon,\delta)}{\sqrt{\lambda}}\|q\|_{L^2_{-\delta}}. \tag{2.8}$$

We shall also need other special solutions to

$$-\Delta u + qu = 0 \quad \text{in} \quad \mathbb{R}^n \tag{2.9}$$

of the form

$$u = e^{\zeta\cdot x}(1+\psi), \tag{2.10}$$

where

$$\zeta\cdot\zeta = 0; \quad \zeta\in\mathbb{C}^n, \tag{2.11}$$

$$\psi\in L^2_\delta; \quad -1<\delta<0, \tag{2.12}$$

and ψ satisfies

$$-\Delta\psi - 2\zeta\cdot\nabla\psi + q\psi = -q. \tag{2.13}$$

We summarize with

Lemma 2.2. *For* $-1<\delta<0$, *there exists* $\varepsilon(\delta)>0$ *such that if*

$$\|(1+|x|)q(x)\|_{L^\infty} < \varepsilon(\delta)|\zeta|, \tag{2.14}$$

there exists a unique solution to (2.9) *of the form* (2.10) *with* (2.12). *In addition,*

$$\|\psi\|_{L^2_\delta} \leqq \frac{C(\varepsilon,\delta)}{|\zeta|}\|q\|_{L^2_{\delta+1}}; \quad -1<\delta<0. \tag{2.15}$$

We shall sketch the proofs to Lemmas 2.1 and 2.2 in an appendix; see also [L–N] for other estimates, which allow more singular potentials. We shall also need

Lemma 2.3. *Let* $q_1, q_2\in L^\infty(\Omega)$, *extended to be zero outside* Ω, *satisfy* (2.7) (*respectively* (2.14)) *and suppose that*

$$\Lambda_{q_1-\lambda} = \Lambda_{q_2-\lambda} \quad (\text{respectively } \Lambda_{q_1} = \Lambda_{q_2}). \tag{2.16}$$

If u_1, u_2 are the unique solutions to (2.1) (respectively (2.9)) of the form (2.2) (respectively (2.10)), then

$$u_1 = u_2 \quad \text{in} \quad \mathbb{R}^n\backslash\Omega.$$

Proof. Let v solve

$$-\Delta v + q_2 v - \lambda v = 0 \quad \text{in} \quad \Omega,$$
$$v|_{\partial\Omega} = u_1|_{\partial\Omega}.$$

Define

$$\omega = \begin{cases} v & \text{for} \quad x\in\Omega \\ u_1 & \text{for} \quad x\in\mathbb{R}^n\backslash\Omega. \end{cases}$$

As a consequence of (2.16), ω and $(\partial\omega/\partial v)$ are continuous across $\partial\Omega$; therefore, ω solves (2.1) in $\mathbb{R}^n$. ω has the appropriate asymptotics at infinity ((2.4) and (2.6)) because u_1 does; so that we may conclude, according to the uniqueness statement in Lemma 2.1, that $\omega = u_2$. Thus $u_2 = u_1$ in $\mathbb{R}^n \setminus \Omega$.

The other case is similar. ■

Proof of Theorem 1.5. Let $m \in \mathbb{R}^n$ be fixed and let

$$\tilde{k} = \tfrac{1}{2}(m + l); \quad m \cdot l = 0,$$
$$k = \tfrac{1}{2}(m - l).$$

Let u_i be as in (2.2) with $q_i (i = 1, 2)$ in place of q (choose $|l|$ so large that (2.7) holds):

$$\int_\Omega e^{i\tilde{k}\cdot x} q_i u_i = \int_\Omega e^{i\tilde{k}\cdot x} (\Delta + \lambda) u_i = \int_{\partial\Omega} e^{i\tilde{k}\cdot x} \left(\frac{\partial u_i}{\partial \nu} - i\tilde{k}\cdot \nu u_i \right) dS,$$

where ν is the outward pointing unit normal and dS is the euclidean surface measure:

$$= \int_{\partial\Omega} e^{i\tilde{k}\cdot x} (\Lambda_{q_i - \lambda} - i\tilde{k}\cdot \nu) u_i \, dS.$$

Now, according to Lemma 2.3,

$$u_1|_{\partial\Omega} = u_2|_{\partial\Omega}$$

and, according to (1.11)

$$\Lambda_{q_1 - \lambda}(u_1|_{\partial\Omega}) = \Lambda_{q_2 - \lambda}(u_2|_{\partial\Omega}),$$

so that we may conclude

$$\int_\Omega e^{i\tilde{k}\cdot x} q_1 u_1 = \int_\Omega e^{i\tilde{k}\cdot x} q_2 u_2.$$

If we now let $|l|$, and hence λ, go to infinity and use (2.8), we obtain

$$\int_\Omega e^{im\cdot x} q_1 = \int_\Omega e^{im\cdot x} q_2.$$

As $m \in \mathbb{R}^n$ was arbitrary, we conclude that

$$q_1 = q_2.$$

To complete the proof of the theorem, we note that (1.10) implies (1.11), as

$$\Lambda_{q-\lambda} = R_{q-\lambda}^{-1} - \alpha(x) I. \quad ■$$

Proof of Theorem 1.6. For fixed $m \in \mathbb{R}^n$, we choose

$$\tilde{\zeta} = \tfrac{1}{2}(k + i(m + e)), \quad \zeta = \tfrac{1}{2}(-k + i(m - e)),$$

where

$$k \cdot e = k \cdot m = e \cdot m = 0, \quad |k| = |m + e| = |m - e|,$$

and compute as before

$$\int_\Omega e^{\tilde{\zeta}\cdot x} q_i u_i = \int_{\partial\Omega} e^{\tilde{\zeta}\cdot x} \left(\frac{\partial u_i}{\partial \nu} - \tilde{\zeta}\cdot \nu u_i \right) dS.$$

We conclude that

$$\int_\Omega e^{\bar{\zeta}\cdot x} q_1 u_1 = \int_\Omega e^{\bar{\zeta}\cdot x} q_2 u_2,$$

and, letting $|e|$, and hence $|k| = |m+e|$, tend to infinity, we conclude from (2.15) that

$$\int_\Omega e^{im\cdot x} q_1 = \int_\Omega e^{im\cdot x} q_2,$$

and hence that $q_1 = q_2$. ■

3. Proof of Theorems 1.3 and 1.4

To make the formal argument following the statement of Theorem 1.5 precise, we shall need two lemmas.

Lemma 3.1. *For m sufficiently large and* $f \in C^\infty(\partial\Omega)$,

$$\left(\frac{d}{d\lambda}\right)^m (R_{q-\lambda}(f)) = \int_{\partial\Omega} r(x,y) f(y)\, dS(y), \tag{3.1}$$

$$\left(\frac{d}{d\lambda}\right)^m (\Lambda_{q-\lambda}(f)) = \int_{\partial\Omega} e(x,y) f(y)\, dS(y), \tag{3.2}$$

where $r(x,y)$ and $e(x,y)$ are the continuous functions in $\bar{\Omega} \times \bar{\Omega}$ given by

$$r(x,y) = \sum_{i=1}^\infty \frac{\psi_i(x)\psi_i(y)}{(\lambda_i - \lambda)^{m+1}} m!, \tag{3.3}$$

$$e(x,y) = \sum_{i=1}^\infty \frac{\frac{\partial\varphi_i}{\partial\nu}(x)\frac{\partial\varphi_i}{\partial\nu}(y)}{(\mu_i - \lambda)^{m+1}} m! \tag{3.4}$$

Proof. Let ω solve

$$(-\Delta + q - \lambda)^{m+1}\omega = 0,$$
$$(-\Delta + q - \lambda)^m \omega|_{\partial\Omega} = m!\, f,$$
$$(-\Delta + q - \lambda)^j \omega|_{\partial\Omega} = 0; \quad 0 \leqq j < m.$$

It is easy to check that

$$\left(\frac{d}{d\lambda}\right)^m (\Lambda_{q-\lambda} f) = \left.\frac{\partial\omega}{\partial\nu}\right|_{\partial\Omega},$$

and that for $x \in \Omega$

$$\omega(x) = -\int_{\partial\Omega} \frac{\partial}{\partial\nu_y}\left\{\frac{d^m}{d\lambda^m} G(x,y,\lambda)\right\} f(y)\, dS(y),$$

so that, for $x_* \in \partial\Omega$

$$\frac{\partial\omega}{\partial\nu}(x_*) = \lim_{x\to x_*} \frac{\partial}{\partial\nu_x} \int_{\partial\Omega} \frac{\partial}{\partial\nu_y}\left\{\frac{d^m}{d\lambda^m} G(x,y,\lambda)\right\} f(y)\, dS(y),$$

where

$$G(x, y, \lambda) = \sum_{i=1}^{\infty} \frac{\varphi_i(x)\varphi_i(y)}{\mu_i - \lambda}.$$

Equations (3.2) and (3.4) will follow as soon as we establish the continuity in $\bar{\Omega} \times \bar{\Omega}$ of

$$e(x, y) = -\left(\frac{d}{d\lambda}\right)^m \frac{\partial}{\partial v_x} \frac{\partial}{\partial v_y} G(x, y, \lambda).$$

To see this, note that the $\varphi_i \in C^\infty(\bar{\Omega})$ and satisfy the straightforward energy estimates

$$\|\varphi_i\|_{H^s(\Omega)} \leqq c_1(q, \Omega)|\mu_i|^{s/2} + c_2(q, \Omega),$$

where c_1 and c_2 depend on q and its derivatives and Ω. A large enough choice of m will therefore assure uniform convergence of (3.4) in $\bar{\Omega} \times \bar{\Omega}$. This proves (3.2) and (3.4); (3.1) and (3.3) are analogous and therefore omitted. ■

Lemma 3.2. *Let* $f \in C^\infty(\partial\Omega)$; $q_1, q_2 \in C^\infty(\bar{\Omega})$; *and* $0 \leqq t < \frac{1}{2}$; *then*

$$\lim_{\lambda \to -\infty} \|(R_{q_1-\lambda} - R_{q_2-\lambda})(f)\|_{H^t(\partial\Omega)} = 0, \tag{3.5}$$

$$\lim_{\lambda \to -\infty} \|(\Lambda_{q_1-\lambda} - \Lambda_{q_2-\lambda})(f)\|_{H^t(\partial\Omega)} = 0. \tag{3.6}$$

Proof. We shall prove (3.6); (3.5) is similar. Let $u_i (i = 1, 2)$ solve

$$(-\Delta + q_i - \lambda)u_i = 0, \quad u_i|_{\partial\Omega} = f. \tag{3.7}$$

Now, $\omega = u_1 - u_2$ solves

$$(-\Delta + q_1 - \lambda)\omega = (q_2 - q_1)u_2, \quad \omega|_{\partial\Omega} = 0,$$

and therefore satisfies the energy estimate

$$\|\omega\|_{H^s(\Omega)} \leqq \frac{C(\Omega) \sup_{x\in\Omega} |q_1 - q_2| \, \|u_2\|_{L^2(\Omega)}}{\left(\inf_{x\in\Omega} |q_1 - \lambda|\right)^{1-s/2}}; \quad 0 \leqq s \leqq 2, \quad \lambda \leqq \lambda_0. \tag{3.8}$$

It follows from (3.7) that

$$\|u_2\|_{L^2(\Omega)} \leqq \frac{C(\Omega) \sup_{x\in\Omega} |q_2 - \lambda| \, \|f\|_{H^{1/2}(\partial\Omega)}}{\inf_{x\in\Omega} |q_2 - \lambda|}; \quad \lambda \leqq \lambda_0. \tag{3.9}$$

For $0 < t < \frac{1}{2}$, combining (3.8) and (3.9) yields

$$\begin{aligned}
\|(\Lambda_{q_1-\lambda} - \Lambda_{q_2-\lambda})f\|_{H^t(\partial\Omega)} &= \left\| \frac{\partial\omega}{\partial v}\bigg|_{\partial\Omega} \right\|_{H^t(\partial\Omega)} \\
&\leqq C(\Omega)\|\omega\|_{H^{t+3/2}(\Omega)} \\
&\leqq \frac{C(\Omega) \sup_{x\in\Omega}|q_2 - \lambda| \sup_{x\in\Omega}|q_1 - q_2| \, \|f\|_{H^{1/2}(\partial\Omega)}}{\left(\inf_{x\in\Omega}|q_1 - \lambda|\right)^{((1-2t)/4)} \left(\inf_{x\in\Omega}|q_2 - \lambda|\right)},
\end{aligned}$$

and the term on the right approaches zero as λ approaches minus infinity.

Proof of Theorems 1.3 and 1.4. Suppose that the hypothesis of Theorem 1.3 holds. Lemma 3.1 implies that $\Lambda_{q_1-\lambda}-\Lambda_{q_2-\lambda}$ is a polynomial in λ, and Lemma 3.2 implies that the polynomial is zero. Hence $\Lambda_{q_1-\lambda}=\Lambda_{q_2-\lambda}$ and we may invoke Theorem 1.5. The proof of Theorem 1.4 is similar. ■

Appendix

In this appendix we sketch a proof of Lemmas 2.1 and 2.2. Lemma 2.1 appears to be a well known consequence of standard arguments from scattering theory (see [A–H] or [A]), so that we shall give only a brief sketch of its proof.

In fact, Lemma 2.1 is easily seen to be a regular perturbation of

Lemma A.1. *Let $f\in L^2_{-\delta}$, $\delta<-\frac{1}{2}$, and $\lambda>\alpha>0$, then there exists a unique L^2_δ λ-outgoing solution to*

$$-\Delta\psi-\lambda\psi=f \quad \text{in} \quad \mathbb{R}^n$$

and

$$\|\psi\|_{L^2_\delta}\leqq\frac{C(\alpha,\delta)}{\sqrt{\lambda}}\|f\|_{L^2_{-\delta}}. \tag{A.1}$$

Sketch of proof. Define

$$\hat{\psi}=\frac{\hat{f}(\xi)}{|\xi|^2-\lambda-i0},$$

where, by definition,

$$\frac{1}{|\xi|^2-\lambda-i0}=\lim_{\varepsilon\downarrow 0^+}\frac{1}{|\xi|^2-\lambda-i\varepsilon}.$$

The estimate (A.1) follows from Theorem 5.1 of [A–H], with a little care taken to keep track of the constant depending on λ. (Note that the $\|\ \ \|_B$ used in [A–H] is strictly weaker than $\|\ \ \|_{L^2_{-\delta}}$ for $\delta<-\frac{1}{2}$). The fact that ψ is the λ-outgoing solution follows from Theorem 7.4 of [A–H], with $Q(x,D)$ taken to be $\partial/\partial r-i\sqrt{\lambda}$. ■

Lemma 2.2 is easily seen to be a regular perturbation of the following which is Proposition 2.1 of [S–U, II]:

Lemma A.2. *Suppose that $\zeta\cdot\zeta=0$, $|\zeta|>B>0$, $-1<\delta<0$, and $f\in L^2_{\delta+1}$; then there exists a unique $\omega\in L^2_\delta$ solving*

$$\Delta\omega+\zeta\cdot\nabla\omega=f;$$

moreover,

$$\|\omega\|_{L^2_\delta}\leqq\frac{C(B,\delta)}{|\zeta|}\|f\|_{L^2_{\delta+1}}.$$

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Note added in proof: We have recently learned that R. G. Novikov has independently obtained results similar to ours (see [N] and further references given there).

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